University of Groningen

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Klüppelberg, Claudia; Mikosch, Thomas

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THE INTEGRATED PERIODOGRAM FOR STABLE PROCESSES

BY CLAUDIA KLÜPPELBERG[1] AND THOMAS MIKOSCH

University of Mainz and University of Groningen

We study the asymptotic behavior of the integrated periodogram for α-stable linear processes. For $\alpha \in (1, 2)$ we prove a functional limit theorem for the integrated periodogram. The limit is an α-stable analogue to the Brownian bridge. We apply our results to investigate some specific goodness-of-fit tests for heavy-tailed linear processes.

1. Introduction. In this paper we consider the linear process

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \qquad t \in \mathbb{Z}, \tag{1.1}$$

where $(Z_t)_{t\in\mathbb{Z}}$ is a noise sequence of iid symmetric α-stable r.v.'s for $\alpha \in (0, 2)$. This implies in particular that Z_t and X_t have infinite variance. In two preceding papers [Klüppelberg and Mikosch (1993, 1994)] we studied the asymptotic behavior of the periodogram

$$I_{n,X}(\lambda) = n^{-2/\alpha}\left|\sum_{t=1}^{n} e^{-i\lambda t} X_t\right|^2, \qquad \lambda \in [-\pi, \pi].$$

The results obtained there indicate that the self-normalized periodogram

$$\widetilde{I}_{n,X}(\lambda) = \frac{I_{n,X}(\lambda)}{\gamma_{n,X}^2}, \qquad \lambda \in [-\pi, \pi],$$

with

$$\gamma_{n,X}^2 = n^{-2/\alpha} \sum_{t=1}^{n} X_t^2$$

behaves very much like the periodogram for finite variance linear processes. More precisely, for all $\alpha \in (0, 2]$, smoothed versions of $\widetilde{I}_{n,X}(\lambda)$ converge in probability to $|\psi(\lambda)|^2/\psi^2$, where

$$|\psi(\lambda)|^2 = \left|\sum_{j=-\infty}^{\infty} \psi_j e^{-ij\lambda}\right|^2, \qquad \lambda \in [-\pi, \pi], \tag{1.2}$$

denotes the power transfer function of the linear filter $(\psi_j)_{j\in\mathbb{Z}}$ and

$$\psi^2 = \sum_{j=-\infty}^{\infty} \psi_j^2.$$

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In the finite variance case $|\psi(\cdot)|^2$ is, up to a constant multiple, the spectral density of the linear process. In the case $\alpha < 2$ this function can naturally not be interpreted as a spectral density of (X_t).

In the present paper we continue our investigation of the spectral analysis of α-stable processes. Our results can be understood as a study of classical (i.e., finite variance) quantities in spectral analysis when some of the innovations Z_t assume very large values. In this sense, the theory given below provides some recommendations on how classical estimators and test statistics have to be modified when large Z_t occur. As appropriate techniques we propose random normalization (we call it *self-normalization*) and random centering.

In the sequel we consider modifications of the integrated periodogram

$$\int_{-\pi}^{x} I_{n,X}(\lambda) f(\lambda)\, d\lambda, \qquad x \in [-\pi, \pi], \tag{1.3}$$

and the corresponding self-normalized version with $I_{n,X}$ replaced by $\widetilde{I}_{n,X}$ for smooth weight functions f.

In the finite variance case, the integrated periodogram serves as an estimate of the spectral distribution function. In analogy to empirical process theory, this suggests building up goodness-of-fit tests of Kolmogorov–Smirnov or Cramér–von Mises type which are based on the integrated periodogram. This idea has been utilized for a long time, for example, by Bartlett, Grenander and Rosenblatt [see Priestley (1981) and Dzhaparidze (1986) and the recent account by Anderson (1993)]. It has been observed by several authors [cf. Anderson (1993) and the references therein] that the asymptotic theory for the standard goodness-of-fit test statistics is actually a consequence of a functional central limit theorem for the integrated periodogram. Thus the asymptotic distribution of the test statistics follows from such a limit theorem in the same way that the asymptotic distribution of Kolmogorov–Smirnov or Cramér–von Mises statistics follows from Donsker's empirical central limit theorem.

This idea also applies in the infinite variance case. To give an example, we shall prove that, for a positive constant C_α and $\alpha \in (1, 2)$,

$$\left(\frac{n}{C_\alpha \log n}\right)^{1/\alpha} \int_{-\pi}^{\cdot} (I_{n,X}(\lambda) - |\psi(\lambda)|^2 T_n)\, d\lambda \ \to_d \ S(\cdot) \equiv 2 \sum_{t=1}^{\infty} Z_t \frac{\sin(\cdot t)}{t} \tag{1.4}$$

in $C[-\pi, \pi]$, the space of continuous functions on $[-\pi, \pi]$ equipped with the uniform topology. Here

$$T_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{I_{n,X}(\lambda)}{|\psi(\lambda)|^2}\, d\lambda.$$

For $\alpha = 2$ the limit process in (1.4) is a Brownian bridge [cf. Hida (1980)], and this is analogous to the result of Anderson (1993) who obtained for the self-normalized integrated periodogram without random centering a Brownian bridge plus an additional Gaussian term. Random centering with (T_n) takes care of this additional term and makes the limit theory for the integrated periodogram very much like empirical process theory. We mention that a similar argument applies to the finite variance case.

From (1.4) we can, for example, immediately derive the limit distribution of the Grenander–Rosenblatt statistic for α-stable (X_t):

$$\left(\frac{n}{C_\alpha \log n}\right)^{1/\alpha} \sup_{-\pi\le x\le\pi}\left|\int_{-\pi}^{x}(I_{n,X}(\lambda)-|\psi(\lambda)|^2 T_n)\,d\lambda\right| \to_d \sup_{-\pi\le x\le\pi}|S(x)|.$$

Tests for α-stable noise or ARMA processes follow easily. This shows the power of the functional central limit theorem (1.4) which reduces the problem of approximating the distribution of the Grenander–Rosenblatt statistic to a study of the properties of the limit process $S(\cdot)$.

Our paper is organized as follows. In Section 2 we introduce some general notation and assumptions. The main theoretical results, in particular the basic functional central limit theorem for the integrated periodogram (Theorem 3.2), are formulated in Section 3. In Section 4 we discuss some goodness-of-fit tests for α-stable (X_t) and illustrate the efficiency of the asymptotic theory by some computer simulations. In particular, we provide tables for the quantiles of the limit distribution of several test statistics. These tables demonstrate the dramatic contrast between the finite and the infinite variance cases. They are followed by Section 5 which contains some auxiliary results for the proofs in Section 6.

2. Assumptions and notation. We consider the moving average process $(X_t)_{t\in\mathbb{Z}}$ defined by (1.1), where $(Z_t)_{t\in\mathbb{Z}}$ is a noise sequence of iid symmetric α-stable r.v.'s for $\alpha\in(0,2)$. This means that the characteristic function of Z_1 is given by

$$Ee^{itZ_1}=e^{-\sigma|t|^\alpha},\qquad t\in\mathbb{R},$$

with a scaling factor $\sigma>0$. For the definition and properties of α-stable r.v.'s, we refer to Feller (1971), Bingham, Goldie and Teugels (1987) or Petrov (1975). We mention that the restriction to symmetric stable r.v.'s is only for ease of presentation. Results can also be derived in the domain of attraction of stable r.v.'s or simply under moment conditions although the proofs then become much more technical [see, e.g., Mikosch, Gadrich, Klüppelberg and Adler (1995)].

In order to guarantee the a.s. absolute convergence of (1.1), we introduce the assumption

$$\sum_{j=-\infty}^{\infty}|j|\,|\psi_j|^\delta<\infty \tag{2.1}$$

for some $\delta<\min(1,\alpha)$. Condition (2.1) is obviously satisfied for every causal invertible ARMA process. Throughout we will also suppose that $|\psi(\lambda)|^2$ is everywhere positive.

The following notation will be used throughout the paper: for any sequence of r.v.'s $(A_t)_{t\in\mathbb{Z}}$,

$$\gamma_{n,A}^2=n^{-2/\alpha}\sum_{t=1}^{n}A_t^2,$$

$$I_{n,A}(\lambda)=n^{-2/\alpha}\left|\sum_{t=1}^{n}A_t\,e^{-i\lambda t}\right|^2,\qquad \lambda\in[-\pi,\pi],$$

$$\widetilde{A}_t = n^{-1/\alpha} A_t / \gamma_{n,A} = A_t \Big/ \left(\sum_{s=1}^{n} A_s^2 \right)^{1/2},$$

$$\widetilde{I}_{n,A}(\lambda) = I_{n,A}(\lambda)/\gamma_{n,A}^2 = \left| \sum_{t=1}^{n} \widetilde{A}_t \, e^{-i\lambda t} \right|^2, \qquad \lambda \in [-\pi, \pi].$$

We introduce

$$y_n = (C_\alpha n \log n)^{1/\alpha},$$

where

$$C_\alpha = \begin{cases} \dfrac{(1-\alpha)\sigma}{2\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \text{if } \alpha \neq 1, \\ \dfrac{\sigma}{\pi}, & \text{if } \alpha = 1, \end{cases}$$

and we set

$$\gamma_{n,A}(k) = y_n^{-1} \sum_{t=1}^{n-|k|} A_t A_{t+|k|}, \qquad |k| \geq 1.$$

In particular, with

$$x_n = \left(\frac{n}{C_\alpha \log n} \right)^{1/\alpha},$$

we obtain that

$$x_n (I_{n,A}(\lambda) - \gamma_{n,A}^2) = 2 \sum_{t=1}^{n-1} \gamma_{n,A}(t) \cos(\lambda t), \qquad n \geq 1.$$

The following proposition is Theorem 3.3 in Davis and Resnick (1986). It is a key result for the present paper. The normalizing sequence $(y_n)_{n \in \mathbb{N}}$ is a consequence of Corollary 2.1 in Rosinski and Woyczynski (1987) and of formula (3.4) in Davis and Resnick (1986).

PROPOSITION 2.1. *Let $(Z_t)_{t \in \mathbb{Z}}$ be a noise sequence satisfying the assumptions above. Let Y_0 be an $\alpha/2$-stable positive r.v. which is independent of $(Z_t)_{t \in \mathbb{Z}}$ and has Laplace transform*

$$E \exp(-r Y_0) = \exp\{-\sigma K_\alpha r^{\alpha/2}\},$$

where $K_\alpha = E|N|^{\alpha/2}$ for an $N(0,2)$-r.v. Then

$$(\gamma_{n,Z}^2, \gamma_{n,Z}(k), \; k = 1, \ldots, m) \to_d (Y_0, Z_k, \; k = 1, \ldots, m), \qquad m \geq 1.$$

For a derivation of the Laplace transform of Y_0, we refer to Klüppelberg and Mikosch (1994), Proposition 3.2.

3. Main results. We start with a consistency result for both the integrated periodogram and the self-normalized integrated periodogram. In the sequel $C[-\pi, \pi]$ denotes the space of the continuous functions on $[-\pi, \pi]$ equipped with the sup-norm.

THEOREM 3.1. *Let $(X_t)_{t\in\mathbb{Z}}$ be defined by* (1.1) *for some $\alpha \in (0, 2)$ and suppose that the coefficients $(\psi_j)_{j\in\mathbb{Z}}$ satisfy condition* (2.1). *Let f be a nonnegative 2π-periodic continuous function such that the Fourier coefficients of $f(\cdot)|\psi(\cdot)|^2$ are absolutely summable. Then*

$$\left(\gamma_{n,X}^2, T_n, \int_{-\pi}^{\cdot} I_{n,X}(\lambda) f(\lambda)\, d\lambda\right) \to_d Y_0\left(\psi^2, 1, \int_{-\pi}^{\cdot} |\psi(\lambda)|^2 f(\lambda)\, d\lambda\right) \tag{3.1}$$

in $\mathbb{R}^1 \times \mathbb{R}^1 \times C[-\pi, \pi]$, where the r.v. Y_0 is defined in Proposition 2.1. *In particular,*

$$\sup_{-\pi\le x\le\pi}\left|\int_{-\pi}^{x}\left(\widetilde{I}_{n,X}(\lambda) - \frac{|\psi(\lambda)|^2}{\psi^2}\right) f(\lambda)\, d\lambda\right| \to_P 0. \tag{3.2}$$

REMARK 3.1. By a Cramér–Wold argument the statement of Theorem 3.1 also holds for general functions f provided both the positive part and the negative part of f satisfy the conditions of Theorem 3.1.

REMARK 3.2. Note that, since $\gamma_{n,X}^2/\psi^2 - T_n \to_P 0$, one can work with T_n instead of $\gamma_{n,X}^2$ in results of type (3.2). This is no longer true if one is interested in the rate of convergence in (3.2) (see Theorem 3.2 below). Indeed, for a finite moving average process, it is not difficult to see that $x_n(\gamma_{n,X}^2/\psi^2 - T_n)$ does not converge weakly to 0.

REMARK 3.3. Note that Theorem 3.1 is valid for all $\alpha \in (0, 2)$. In the classical case where Z_1 has a second moment, the result remains formally true with Y_0 replaced by $2\sigma^2$. This is a consequence of the law of large numbers. Thus, if we use the self-normalized version $\widetilde{I}_{n,X}$, we are not able to distinguish between an infinite variance α-stable process $(X_t)_{t\in\mathbb{Z}}$ and a process with finite second moment. This can be interpreted as a robustness property of the self-normalized periodogram which has intuitively been known for a long time [see Priestley (1981)]. On the other hand, (3.1) and (3.2) show the profound difference in the limit behavior of the periodogram and the self-normalized version. The contrast between the infinite and the finite variance case again manifests in the rates of convergence in Theorem 3.2 below.

EXAMPLE 3.1. For $f(\cdot) = |\psi(\cdot)|^{-2}$ we obtain from Theorem 3.1 that

$$\sup_{-\pi\le x\le\pi}\left|\int_{-\pi}^{x}\left(\frac{\widetilde{I}_{n,X}(\lambda)}{|\psi(\lambda)|^2} - \frac{1}{\psi^2}\right) d\lambda\right| \to_P 0.$$

This result holds for any causal invertible ARMA process.

EXAMPLE 3.2. The function $f \equiv 1$ satisfies the assumptions of Theorem 3.1. Therefore,

$$\sup_{-\pi \le x \le \pi} \left| \int_{-\pi}^{x} \left(\widetilde{I}_{n,X}(\lambda) - \frac{|\psi(\lambda)|^2}{\psi^2} \right) d\lambda \right| \to_P 0.$$

THEOREM 3.2. *Let $(X_t)_{t\in\mathbb{Z}}$ be defined by* (1.1) *with coefficients $(\psi_j)_{j\in\mathbb{Z}}$ satisfying* (2.1) *and suppose that $\alpha \in (1,2)$. Furthermore, assume that f is defined on $[-\pi,\pi]$ such that $g(\cdot) = f(\cdot)|\psi(\cdot)|^2$ is continuously differentiable. Then*

$$\begin{aligned} &\left(\gamma_{n,X}^2, T_n, x_n \int_{-\pi}^{\cdot} (I_{n,X}(\lambda) - |\psi(\lambda)|^2 T_n) f(\lambda)\, d\lambda \right) \\ &\quad \to_d \left(Y_0 \psi^2, Y_0, 2\sum_{t=1}^{\infty} Z_t \int_{-\pi}^{\cdot} g(\lambda) \cos(t\lambda)\, d\lambda \right) \\ &\quad =_d \left(Y_0 \psi^2, Y_0, g(\cdot) S(\cdot) - \int_{-\pi}^{\cdot} g'(\lambda) S(\lambda)\, d\lambda \right) \end{aligned} \tag{3.3}$$

in $\mathbb{R}^1 \times \mathbb{R}^1 \times C[-\pi,\pi]$, where

$$S(\cdot) = 2\sum_{t=1}^{\infty} \frac{\sin(\cdot t)}{t} Z_t \tag{3.4}$$

and Y_0 is independent of $(Z_t)_{t\in\mathbb{Z}}$ with the same distribution as in Proposition 2.1.

REMARK 3.4. Notice that the representation (3.4) is analogous to the Lévy–Cieselski or Paley–Wiener representation of a Brownian bridge [e.g., Hida (1980)]. Indeed, if $(Z_t)_{t\in\mathbb{Z}}$ is iid Gaussian white noise, then (3.4) represents a Brownian bridge.

REMARK 3.5. The series S is a random Fourier series with α-stable coefficients and as such it is a harmonizable stable process with discrete spectral measure. As a limit process in $C[-\pi,\pi]$, it has continuous sample paths. For its tails we conclude that

$$P\Big(\sup_{-\pi \le x \le \pi} S(x) > t \Big) = O(t^{-\alpha}), \qquad t \to \infty.$$

Figure 1 shows simulated sample paths of the process $(S(x))_{0\le x\le \pi}$ for different values of α. Some of them look surprisingly regular, at a first sight almost deterministic like slightly perturbed sine curves. For comparison the first picture shows a simulated Brownian bridge, that is, a path of S for $\alpha = 2$. The sample paths for $\alpha < 2$ behave completely differently. According to our limited experience from simulations, "deterministic" behavior appears more often with decreasing α. The sine shape can be well explained: since the distribution tail of the r.v. Z_1 is very heavy, some of the Z_t will be huge compared with the others. Thus the contribution of the sine function $Z_t(\sin(\cdot t)/t)$ to the series S is larger the larger the value of Z_t and the smaller t.

We consider some particular cases of Theorem 3.2. First we choose $f \equiv 1$.

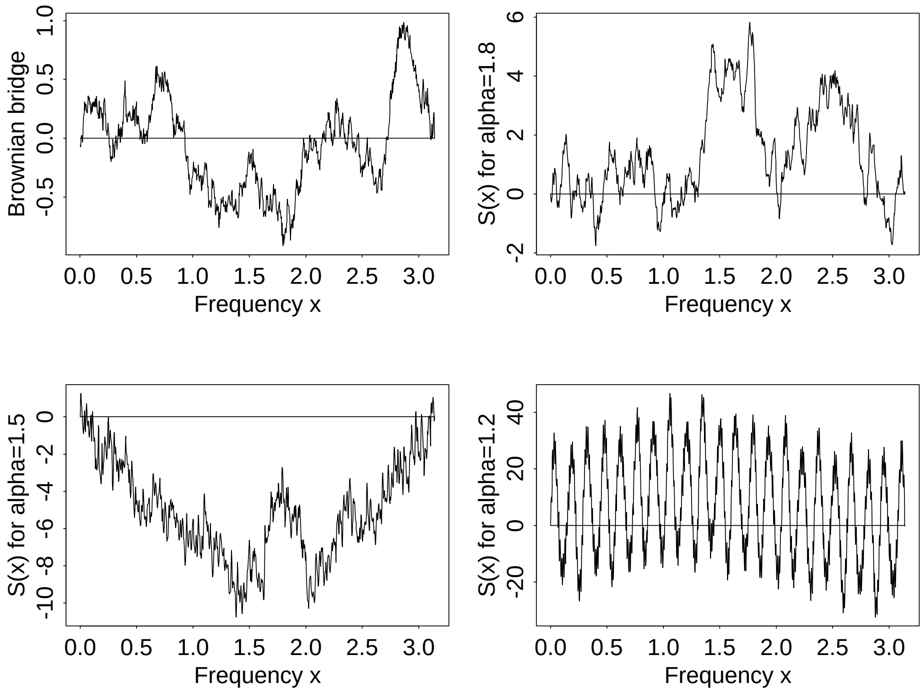

FIG. 1. *Sample paths of the process S on $[0,\pi]$ with α-stable noise.*

COROLLARY 3.3. *Under the assumptions of Theorem* 3.2, *we obtain that*

$$x_n \int_{-\pi}^{\cdot} \Big(I_{n,X}(\lambda) - |\psi(\lambda)|^2 T_n \Big)\, d\lambda \to_d 2 \sum_{t=1}^{\infty} Z_t \int_{-\pi}^{\cdot} |\psi(\lambda)|^2 \cos(t\lambda)\, d\lambda,$$

$$x_n \int_{-\pi}^{\cdot} \bigg(\frac{I_{n,X}(\lambda)}{T_n} - |\psi(\lambda)|^2 \bigg)\, d\lambda \to_d 2 \sum_{t=1}^{\infty} \frac{Z_t}{Y_0} \int_{-\pi}^{\cdot} |\psi(\lambda)|^2 \cos(t\lambda)\, d\lambda.$$

Now we choose $f(\cdot) = |\psi(\cdot)|^{-2}$.

COROLLARY 3.4. *Under the assumptions of Theorem* 3.2, *we obtain that*

$$x_n \int_{-\pi}^{\cdot} \bigg(\frac{I_{n,X}(\lambda)}{|\psi(\lambda)|^2} - T_n \bigg)\, d\lambda \to_d S(\cdot),$$

$$x_n \int_{-\pi}^{\cdot} \bigg(\frac{I_{n,X}(\lambda)}{T_n |\psi(\lambda)|^2} - 1 \bigg)\, d\lambda \to_d \frac{S(\cdot)}{Y_0}.$$

REMARK 3.6. Corollary 3.4 is analogous to the finite variance case where the limit process is a Brownian bridge [see Grenander and Rosenblatt (1957), Priestley (1981) and Dzhaparidze (1986)].

In the case $\alpha \in (0, 1]$ we cannot expect a result which is similar to Theorem 3.2. For example, note that

$$S(x) =_d 2Z_1 \bigg(\sum_{t=1}^{\infty} \bigg| \frac{\sin(xt)}{t} \bigg|^{\alpha} \bigg)^{1/\alpha},$$

where the series on the right-hand side diverges except for $x = 0, \pi, -\pi$. Nevertheless, in some cases it is possible to determine the limit law of the integrated periodogram for a fixed frequency. This follows from the next result.

PROPOSITION 3.5. *Let $(X_t)_{t\in\mathbb{Z}}$ be a linear process as defined in* (1.1) *with coefficients $(\psi_j)_{j\in\mathbb{Z}}$ satisfying* (2.1) *and suppose that $\alpha \in (0, 2)$. Furthermore, assume that f is defined on $[-\pi, \pi]$ such that $g(\cdot) = f(\cdot)|\psi(\cdot)|^2$ is continuous and*

$$\sum_{t=1}^{\infty} \bigg| \int_{-\pi}^{x} g(\lambda) \cos(t\lambda) d\lambda \bigg|^{\mu} < \infty \tag{3.5}$$

for some $x \in [-\pi, \pi]$ and some $0 < \mu < \alpha$. Then

$$\begin{aligned} &\bigg(\gamma_{n,X}^2, T_n, x_n \int_{-\pi}^{x} (I_{n,X}(\lambda) - |\psi(\lambda)|^2 T_n) f(\lambda)\, d\lambda \bigg) \\ &\quad \to_d \bigg(Y_0 \psi^2, Y_0, 2 \sum_{t=1}^{\infty} Z_t \int_{-\pi}^{x} g(\lambda) \cos(t\lambda)\, d\lambda \bigg) \\ &\quad =_d \bigg(Y_0 \psi^2, Y_0, 2Z_1 \bigg(\sum_{t=1}^{\infty} \bigg| \int_{-\pi}^{x} g(\lambda) \cos(t\lambda)\, d\lambda \bigg|^{\alpha} \bigg)^{1/\alpha} \bigg), \end{aligned}$$

with Y_0 independent of $(Z_t)_{t\in\mathbb{Z}}$ and with the same distribution as in Proposition 2.1.

4. Goodness-of-fit tests for α-stable processes. In this section we review some specific goodness-of-fit tests and study their asymptotic behavior under the assumption of heavy-tailedness. Note that all test statistics are continuous functionals of the integrated periodogram. Hence their weak limits are immediate consequences of Theorem 3.2 (see also Corollaries 3.3 and 3.4).

The following results hold under the assumptions of Theorem 3.2 for the linear process (1.1). They are modifications of tests which are well studied in the finite variance case [e.g., Priestley (1981), subsection 6.2.6].

1. Grenander and Rosenblatt's test for the nonnormalized integrated spectrum:

$$\begin{aligned} &x_n \sup_{-\pi \le x \le \pi} \bigg| \int_{-\pi}^{x} (I_{n,X}(\lambda) - |\psi(\lambda)|^2 T_n)\, d\lambda \bigg| \\ &\qquad \to_d \sup_{-\pi \le x \le \pi} \bigg| 2 \sum_{t=1}^{\infty} Z_t \int_{-\pi}^{x} |\psi(\lambda)|^2 \cos(\lambda t)\, d\lambda \bigg|. \end{aligned}$$

Test for α-stable noise:

$$x_n \sup_{-\pi \le x \le \pi} \left| \int_{-\pi}^{x} (I_{n,X}(\lambda) - T_n)\, d\lambda \right| \to_d \sup_{-\pi \le x \le \pi} |S(x)|.$$

2. Bartlett's test for the self-normalized integrated spectrum:

$$x_n \sup_{-\pi \le x \le \pi} \left| \int_{-\pi}^{x} \left(\widetilde{I}_{n,X}(\lambda) - \frac{|\psi(\lambda)|^2 T_n}{\gamma_{n,X}^2} \right) d\lambda \right|$$
$$\to_d \frac{1}{Y_0 \psi^2} \sup_{-\pi \le x \le \pi} \left| 2 \sum_{t=1}^{\infty} Z_t \int_{-\pi}^{x} |\psi(\lambda)|^2 \cos(\lambda t)\, d\lambda \right|.$$

Test for α-stable noise:

$$x_n \sup_{-\pi \le x \le \pi} \left| \int_{-\pi}^{x} \left(\widetilde{I}_{n,X}(\lambda) - \frac{T_n}{\gamma_{n,X}^2} \right) d\lambda \right| \to_d \sup_{-\pi \le x \le \pi} \frac{|S(x)|}{\psi^2 Y_0}.$$

3. Bartlett's T_p test:

$$x_n \sup_{-\pi \le x \le \pi} \left| \int_{-\pi}^{x} \left(\frac{I_{n,X}(\lambda)}{|\psi(\lambda)|^2 T_n} - 1 \right) d\lambda \right| \to_d \sup_{-\pi \le x \le \pi} \frac{|S(x)|}{Y_0}.$$

4. ω^2-statistic or Cramér–von Mises test:

$$x_n^2 \int_{-\pi}^{\pi} \left(\int_{-\pi}^{x} \left(\frac{I_{n,X}(\lambda)}{|\psi(\lambda)|^2} - T_n \right) d\lambda \right)^2 dx \to_d \int_{-\pi}^{\pi} S^2(x)\, dx = 4\pi \sum_{t=1}^{\infty} \frac{Z_t^2}{t^2}.$$

5. ω^2-test or Cramér–von Mises test with normalization T_n for the integrated spectrum:

$$x_n^2 \int_{-\pi}^{\pi} \left(\int_{-\pi}^{x} \left(\frac{\widetilde{I}_{n,X}(\lambda)}{|\psi(\lambda)|^2 T_n} - 1 \right) d\lambda \right)^2 dx \to_d \frac{1}{Y_0^2} \int_{-\pi}^{\pi} S^2(x)\, dx = \frac{4\pi}{Y_0^2} \sum_{t=1}^{\infty} \frac{Z_t^2}{t^2}.$$

The above results can immediately be compared with the finite variance case. In the classical situation the normalizing constants x_n are of the order $\sqrt{n}$, and S represents a Brownian bridge [see Hida (1980), Priestley (1981) and Dzhaparidze (1986)]. Formally, the limits in the cases $\alpha \in (1, 2)$ and $\alpha = 2$ are the same if we replace Y_0 by $2\sigma^2$ for $\alpha = 2$.

The rate of convergence in the α-stable case compares favorably with the finite variance case. This seems to be the rule for the limit theory of α-stable time series, in the time domain, in the frequency domain, but also in estimation theory for linear processes. We refer to Mikosch, Gadrich, Klüppelberg and Adler (1995) for a discussion of this phenomenon and more references. However, the limit distributions of the above statistics are much less common than in the finite variance case. Their study is by no means trivial.

The distribution of the quadratic functional $\sum_{t=1}^{\infty}(Z_t^2/t^2)$ is characterized by the relation

$$\sum_{t=1}^{\infty} \frac{Z_t^2}{t^2} =_d Y_0 \left(K_\alpha^{-1} \sum_{t=1}^{\infty} \left| \frac{N_t}{t} \right|^{\alpha} \right)^{2/\alpha}.$$

Here Y_0 is $\alpha/2$-stable as defined in Proposition 2.1 and $(N_t)_{t\in\mathbb{Z}}$ are iid $N(0,2)$-Gaussian r.v.'s independent of Y_0. Indeed, for $(N_t)_{t\in\mathbb{Z}}$ independent of $(Z_t)_{t\in\mathbb{Z}}$, we have, for positive λ,

$$\begin{aligned}
E\exp\left\{-\lambda\sum_{t=1}^{\infty}\frac{Z_t^2}{t^2}\right\} &= E\exp\left\{i\lambda^{1/2}\sum_{t=1}^{\infty}\frac{Z_tN_t}{t}\right\}\\
&= E\exp\left\{i\lambda^{1/2}Z_1\left(\sum_{t=1}^{\infty}\left|\frac{N_t}{t}\right|^{\alpha}\right)^{1/\alpha}\right\}\\
&= E\exp\left\{-\sigma\lambda^{\alpha/2}K_\alpha\left(K_\alpha^{-1}\sum_{t=1}^{\infty}\left|\frac{N_t}{t}\right|^{\alpha}\right)\right\}\\
&= E\exp\left\{-\lambda Y_0\left(K_\alpha^{-1}\sum_{t=1}^{\infty}\left|\frac{N_t}{t}\right|^{\alpha}\right)^{2/\alpha}\right\}.
\end{aligned}$$

In particular, let Y_0 and Y_1 be iid and independent of $(N_t)_{t\in\mathbb{Z}}$. Then we conclude that

$$x_n^2\int_{-\pi}^{\pi}\left(\int_{-\pi}^{x}\left(\frac{\widetilde{I}_{n,X}(\lambda)}{|\psi(\lambda)|^2}-\frac{1}{\psi^2}\right)d\lambda\right)^2dx\to_d\frac{4\pi Y_0}{\psi^4Y_1^2}\left(K_\alpha^{-1}\sum_{t=1}^{\infty}\left|\frac{N_t}{t}\right|^{\alpha}\right)^{2/\alpha}.$$

We conclude this section with a short simulation study in order to show the efficiency of the asymptotic theory given above. The study was carried out in S-Plus on a SUN workstation and we chose the parameters (sample size, number of discretization points and terms of the series) within the limits of available memory and computer time. The theoretical evaluation of the distributions of continuous functionals of $S(\cdot)$ seems very difficult and calls for further research. We restrict ourselves to some simulation studies in order to show that the proposed methods already work for medium sample sizes n between 200 and 400. For obvious symmetry reasons, we only consider the process $S(\cdot)$ and its functionals on the interval $[0,\pi]$. The dramatic contrast between the cases $\alpha<2$ and the Brownian bridge, visualized in Figure 1, is also well illustrated by the following tables. In Table 1 we give the quantiles of the limit distribution

$$A_\alpha=\sup_{0\le x\le\pi}|S(x)|$$

for statistics of Grenander–Rosenblatt type with α-stable innovations (Z_t). For reasons of comparability, we include also the case $\alpha=2$ which corresponds to a Brownian bridge. The quantiles for the Brownian bridge were calculated from the corresponding ones of the absolute supremum functional of the standard Brownian bridge on $[0,1]$ provided in Smirnov (1948); see Shorack and Wellner (1986), page 143. The quantiles for $\alpha=1.2$ and $\alpha=1.8$ are the empirical ones from 800 independent simulations of A_α with 700 terms of the series S.

In Table 2 we give the quantiles of the limit distribution

$$B_\alpha=\int_0^{\pi}S^2(x)\,dx$$

TABLE 1
Quantiles of the absolute supremum functional of S on $[0, \pi]$

Quantile	$\boldsymbol{A_2}$	$\boldsymbol{A_{1.8}}$	$\boldsymbol{A_{1.2}}$
0.05	2.31	3.43	6.99
0.10	2.53	3.82	7.82
0.15	2.71	4.04	8.61
0.20	2.89	4.26	9.36
0.25	3.02	4.50	10.00
0.30	3.15	4.73	10.80
0.35	3.29	4.97	11.51
0.40	3.42	5.19	12.28
0.45	3.55	5.40	13.17
0.50	3.69	5.70	14.23
0.55	3.82	5.96	15.21
0.60	4.00	6.21	16.25
0.65	4.13	6.51	17.60
0.70	4.31	6.74	19.09
0.75	4.53	7.17	20.90
0.80	4.75	7.52	23.87
0.85	5.06	8.08	30.00
0.90	5.42	8.79	39.44
0.91	5.55	9.24	44.22
0.92	5.64	9.56	48.16
0.93	5.78	9.84	51.53
0.94	5.91	10.11	58.77
0.95	6.04	10.68	61.98
0.96	6.22	11.38	73.95
0.97	6.44	12.39	85.71
0.98	6.75	14.07	103.4
0.99	7.20	17.69	180.3

of statistics of ω^2-type with α-stable innovations (Z_t). The quantiles for the Brownian bridge were calculated from the corresponding ones of the ω^2-functional of the standard Brownian bridge on $[0, 1]$ provided in Anderson and Darling (1952); see Shorack and Wellner (1986), page 147. The quantiles for $\alpha = 1.2$ and $\alpha = 1.8$ are the empirical ones from 50,000 independent simulations of B_α with 600 terms in the series representation.

In a simulation study we convinced ourselves that the goodness-of-fit tests work well in the heavy-tailed case for medium sample sizes of 200 to 400. In a first experiment we considered the Grenander–Rosenblatt statistic for 1.8-stable noise modified for the interval $[0, \pi]$; that is,

$$A(n) = \sup_{0 \le x \le \pi} \left| \int_{-\pi}^{x} (I_{n,Z}(\lambda) - T_n)\, d\lambda \right|.$$

In Figure 2, for a realization (Z_t) of 1.8-stable noise, the values $A(n)$ are plotted against n (thin line). For comparison, the values $10.68/x_n$ (thick line) are also plotted. The value 10.68 stands for the 0.95 quantile of the distribution of $A_{1.8}$ (see Table 1).

TABLE 2
Quantiles of the ω^2-functional of S on $[0, \pi]$

Quantile	B_2	$B_{1.8}$	$B_{1.2}$
0.05	2.27	10.66	34.94
0.10	2.85	13.58	46.97
0.15	3.36	16.06	58.34
0.20	3.86	18.43	70.05
0.25	4.36	20.88	83.15
0.30	4.87	23.38	98.24
0.35	5.42	26.11	116.1
0.40	6.01	29.06	135.5
0.45	6.66	32.31	159.6
0.50	7.37	36.07	190.6
0.55	8.17	40.21	228.5
0.60	9.09	45.22	278.3
0.65	10.16	50.87	350.9
0.70	11.43	57.75	447
0.75	12.98	66.54	602
0.80	14.96	78.50	852
0.85	17.61	95.99	1,334
0.90	21.54	125.3	2,605
0.91	22.58	134.5	3,056
0.92	23.77	144.6	3,694
0.93	25.13	157.3	4,520
0.94	26.71	172.6	5,873
0.95	28.61	195.4	7,799
0.96	30.96	224.9	11,109
0.97	34.03	281.3	17,955
0.98	38.43	383.7	36,548
0.99	46.10	747.6	110,210

In a second experiment we simulated realizations of an AR(1) process $X_t = 0.5X_{t-1}+Z_t$ with 1.8-stable noise (Z_t) and calculated the ω^2-statistic modified for the interval $[0, \pi]$; that is,

$$B(n) = \int_0^{\pi}\left(\int_{-\pi}^{x}\left(\frac{I_{n,X}(\lambda)}{|\psi(\lambda)|^2} - T_n\right)d\lambda\right)^2 dx.$$

In Figure 3 we plotted the values $B(n)$ (thin line) against $195.4/x_n^2$ (thick line) where 195.4 is the 0.95 quantile of $B_{1.8}$ (see Table 2).

5. Auxiliary results. The following statement is part of Proposition 4.3 in Davis and Resnick (1986).

LEMMA 5.1. *Under* (1.1) *and* (2.1),

$$\gamma_{n,X}^2 - \psi^2\gamma_{n,Z}^2 \to_P 0.$$

We will frequently make use of the following decomposition of the periodogram [see Klüppelberg and Mikosch (1993), Proposition 2.1].

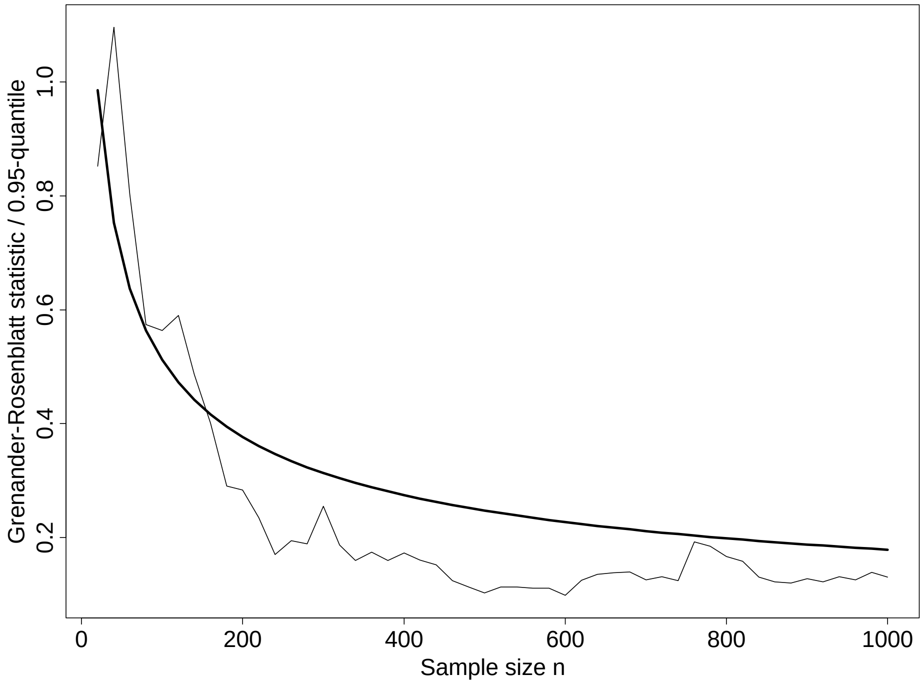

FIG. 2. *The Grenander–Rosenblatt statistic* $A(n)$ (*thin line*) *and the* (0.95 *quantile of* $A_{1.8})/x_n$ (*thick line*).

LEMMA 5.2. *Under* (1.1) *and* (2.1),

$$I_{n,X}(\lambda) = |\psi(\lambda)|^2 \, I_{n,Z}(\lambda) + R_n(\lambda), \qquad -\pi < \lambda \le \pi,$$

where

$$R_n(\lambda) = \psi(\lambda)\, J_n(\lambda)\, Y_n(-\lambda) + \psi(-\lambda)\, J_n(-\lambda)\, Y_n(\lambda) + \left|Y_n(\lambda)\right|^2,$$

$$J_n(\lambda) = n^{-1/\alpha} \sum_{t=1}^{n} Z_t \, e^{-i\lambda t},$$

$$Y_n(\lambda) = n^{-1/\alpha} \sum_{j=-\infty}^{\infty} \psi_j \, e^{-i\lambda j} \, U_{nj}(\lambda),$$

$$U_{nj}(\lambda) = \sum_{t=1-j}^{n-j} Z_t \, e^{-i\lambda t} - \sum_{t=1}^{n} Z_t \, e^{-i\lambda t}.$$

From the proof of Lemma 9.2 in Mikosch, Gadrich, Klüppelberg and Adler (1995), we conclude the following result [which has been proved there for

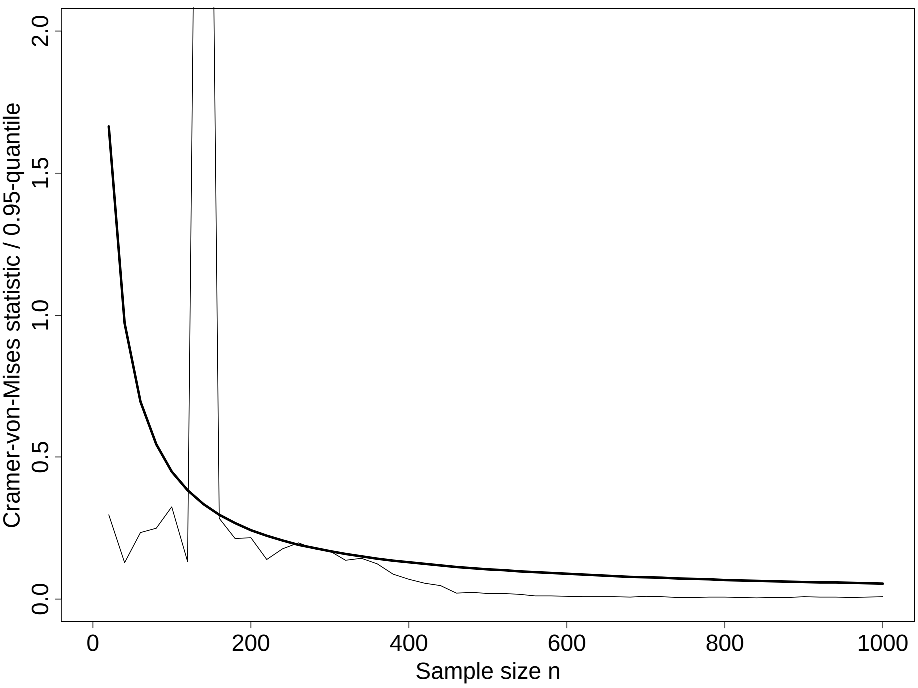

FIG. 3. *The Cramér–von Mises statistic* $B(n)$ (*thin line*) *and the* (0.95 *quantile of* $B_{1.8}$)$/x_n^2$ (*thick line*).

ARMA processes, but a careful study of the proof shows that it remains valid for more general processes $(X_t)_{t\in\mathbb{Z}}$]:

LEMMA 5.3. *Under* (1.1) *and* (2.1), *the remainder* R_n *defined in Lemma* 5.2 *satisfies the relation*

$$x_n \int_{-\pi}^{\pi} |R_n(\lambda)|\, d\lambda \to_P 0.$$

From Lemmas 5.2 and 5.3 we conclude the following.

LEMMA 5.4. *Under* (1.1) *and* (2.1),

$$x_n(T_n - \gamma_{n,Z}^2) \to_P 0.$$

We also need the following elementary tool.

LEMMA 5.5. *Let* G_1, G_2 *be jointly Gaussian with zero mean and identical variances. Then*

$$G_1G_2 =_d \tfrac{1}{2}\mathrm{Var}(G_1)(N_1^2 - N_2^2) + \tfrac{1}{2}\mathrm{Cov}(G_1, G_2)(N_1^2 + N_2^2),$$

where N_1, N_2 *are iid* $N(0,1)$-*r.v.'s*.

PROOF. The quadratic form

$$G_1G_2 = \tfrac{1}{2}(G_1G_2 + G_2G_1) = G^T A G,$$

with

$$A = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

can be written as $z_1N_1^2 + z_2N_2^2$ for iid standard Gaussian N_1, N_2 and for the eigenvalues z_1, z_2 of the matrix ΣA, where Σ is the covariance matrix of G. The eigenvalues of ΣA are

$$z_1 = \tfrac{1}{2}(\mathrm{Var}(G_1) + \mathrm{Cov}(G_1, G_2))$$

and

$$z_2 = \tfrac{1}{2}(\mathrm{Cov}(G_1, G_2) - \mathrm{Var}(G_1)).$$

Hence

$$z_1N_1^2 + z_2N_2^2$$

is distributed as

$$\tfrac{1}{2}\mathrm{Var}(G_1)(N_1^2 - N_2^2) + \tfrac{1}{2}\mathrm{Cov}(G_1, G_2)(N_1^2 + N_2^2). \qquad \square$$

6. Proofs. Throughout, c will denote a generic positive constant which may change from formula to formula or from line to line.

PROOF OF THEOREM 3.1. We define $g^2(\cdot) = f(\cdot)|\psi(\cdot)|^2$. In view of Lemmas 5.1 to 5.4 we have, uniformly for $x \in [-\pi, \pi]$,

$$\begin{aligned}&\left(\gamma_{n,X}^2, T_n, \int_{-\pi}^{x} I_{n,X}(\lambda) f(\lambda)\, d\lambda\right)\\ &\qquad = (1 + o_P(1))\left(\psi^2\gamma_{n,Z}^2, \gamma_{n,Z}^2, \int_{-\pi}^{x} I_{n,Z}(\lambda) g^2(\lambda)\, d\lambda\right).\end{aligned}$$

Here we also used the boundedness of f and $|\psi(\lambda)|^2$. We define

$$\begin{aligned}M_n(\cdot) &= \int_{-\pi}^{\cdot} I_{n,Z}(\lambda) g^2(\lambda)\, d\lambda\\ &= \int_{-\pi}^{\cdot} ((\mathrm{Re}(J_n(\lambda)))^2 + (\mathrm{Im}(J_n(\lambda)))^2) g^2(\lambda)\, d\lambda\end{aligned}$$

and show the fidi-convergence

$$(\gamma_{n,Z}^2, M_n(\cdot)) \to_d Y_0\left(1, \int_{-\pi}^{\cdot} g^2(\lambda)\, d\lambda\right). \tag{6.1}$$

We restrict ourselves to the case of two dimensions. Suppose $-\pi < x < y \le \pi$. Let B_i, $i = 1, 2, 3, 4$, be independent Brownian motions on $[-\pi, \pi]$, $(N_t)_{t\in\mathbb{Z}}$

be iid $N(0,1)$-r.v.'s and let (Z_t), (B_i) and (N_i) be independent. Then, for real r_i, $i=1,2,3$,

$$
\begin{aligned}
&E\exp\left\{-\frac{r_1^2}{2}M_n(x)-\frac{r_2^2}{2}M_n(y)-\frac{r_3^2}{2}\gamma_{n,Z}^2\right\}\\
&\quad= E\exp\left\{-\frac{r_1^2}{2}\int_{-\pi}^{x}((\mathrm{Re}(J_n(\lambda)))^2+(\mathrm{Im}(J_n(\lambda)))^2)g^2(\lambda)\,d\lambda\right.\\
&\qquad\qquad\left.-\frac{r_2^2}{2}\int_{-\pi}^{y}((\mathrm{Re}(J_n(\lambda)))^2+(\mathrm{Im}(J_n(\lambda)))^2)g^2(\lambda)\,d\lambda-\frac{r_3^2}{2}\gamma_{n,Z}^2\right\}\\
&\quad= E\exp\left\{ir_1\left(\int_{-\pi}^{x}\mathrm{Re}(J_n(\lambda))g(\lambda)\,dB_1(\lambda)+\int_{-\pi}^{x}\mathrm{Im}(J_n(\lambda))g(\lambda)\,dB_2(\lambda)\right)\right.\\
&\qquad\qquad+ir_2\left(\int_{-\pi}^{y}\mathrm{Re}(J_n(\lambda))g(\lambda)\,dB_3(\lambda)+\int_{-\pi}^{y}\mathrm{Im}(J_n(\lambda))g(\lambda)\,dB_4(\lambda)\right)\\
&\qquad\qquad\left.+ir_3n^{-1/\alpha}\sum_{t=1}^{n}Z_tN_t\right\}\\
&\quad= E\exp\left\{in^{-1/\alpha}\sum_{t=1}^{n}Z_t\left(r_1\int_{-\pi}^{x}\cos(\lambda t)g(\lambda)\,dB_1(\lambda)\right.\right.\\
&\qquad\qquad+r_1\int_{-\pi}^{x}\sin(\lambda t)g(\lambda)\,dB_2(\lambda)\\
&\qquad\qquad+r_2\int_{-\pi}^{y}\cos(\lambda t)g(\lambda)\,dB_3(\lambda)\\
&\qquad\qquad\left.\left.+r_2\int_{-\pi}^{y}\sin(\lambda t)g(\lambda)\,dB_4(\lambda)+r_3N_t\right)\right\}\\
&\quad= E\exp\left\{-\frac{\sigma}{n}\sum_{t=1}^{n}\left|r_1\int_{-\pi}^{x}\cos(\lambda t)g(\lambda)\,dB_1(\lambda)+r_1\int_{-\pi}^{x}\sin(\lambda t)g(\lambda)\,dB_2(\lambda)\right.\right.\\
&\qquad\qquad+r_2\int_{-\pi}^{y}\cos(\lambda t)g(\lambda)\,dB_3(\lambda)\\
&\qquad\qquad\left.\left.+r_2\int_{-\pi}^{y}\sin(\lambda t)g(\lambda)\,dB_4(\lambda)+r_3N_t\right|^{\alpha}\right\}\\
&\quad=: E\exp\left\{-\frac{\sigma}{n}\sum_{t=1}^{n}|G_t|^{\alpha}\right\}.
\end{aligned}
$$

The r.v.'s

$$
\begin{aligned}
G_t &= r_1\int_{-\pi}^{x}\cos(\lambda t)g(\lambda)\,dB_1(\lambda)+r_1\int_{-\pi}^{x}\sin(\lambda t)g(\lambda)\,dB_2(\lambda)\\
&\quad+r_2\int_{-\pi}^{y}\cos(\lambda t)g(\lambda)\,dB_3(\lambda)+r_2\int_{-\pi}^{y}\sin(\lambda t)g(\lambda)\,dB_4(\lambda)+r_3N_t
\end{aligned}
$$

are Gaussian with expectation 0 and covariances (δ_{ts} is the Kronecker symbol)

$$
\begin{aligned}
&\mathrm{Cov}(G_t,G_s)\\
&\quad= r_1^2\int_{-\pi}^{x}\cos(\lambda(t-s))g^2(\lambda)\,d\lambda+r_2^2\int_{-\pi}^{y}\cos(\lambda(t-s))g^2(\lambda)\,d\lambda+r_3^2\delta_{ts}.
\end{aligned}
$$

We have for a standard Gaussian r.v. N_1 that

$$E|G_t|^\alpha = E|N_1|^\alpha \left(r_1^2 \int_{-\pi}^{x} g^2(\lambda)\,d\lambda + r_2^2 \int_{-\pi}^{y} g^2(\lambda)\,d\lambda + r_3^2 \right)^{\alpha/2}$$
$$= E|N_1|^\alpha (\mathrm{Var}(G_1))^{\alpha/2}$$

independent of t. Applying Lemma 5.5, we conclude that $G_t G_s$ is distributed as

$$\tfrac{1}{2}\mathrm{Var}(G_1)(N_1^2 - N_2^2) + \tfrac{1}{2}\mathrm{Cov}(G_t, G_s)(N_1^2 + N_2^2).$$

In order to show that

$$n^{-1} \sum_{t=1}^{n} |G_t|^\alpha \to_P E|G_1|^\alpha$$

holds, it suffices to show that

$$\mathrm{var}\left(n^{-1} \sum_{t=1}^{n} |G_t|^\alpha \right) = E\left(n^{-1} \sum_{t=1}^{n} (|G_t|^\alpha - E|G_1|^\alpha) \right)^2 \to 0.$$

Hence it suffices to show that the following expression converges to 0:

$$\frac{1}{n^2} \sum_{t \neq s} (E|G_t G_s|^\alpha - (E|G_1|^\alpha)^2)$$
$$= \frac{1}{n^2} \sum_{t \neq s} \left(E\left| \frac{1}{2}\mathrm{Var}(G_1)(N_1^2 - N_2^2) + \frac{1}{2}\mathrm{Cov}(G_t, G_s)(N_1^2 + N_2^2) \right|^\alpha \right.$$
$$\left. - (E|N_1|^\alpha)^2 (\mathrm{Var}(G_1))^\alpha \right).$$

Notice that $N_1^2 - N_2^2$ has the same distribution as $2N_1N_2$. Using the continuity of the l^α-seminorm, it suffices to show that

$$n^{-2} \sum_{t \neq s} E|\mathrm{Cov}(G_t, G_s)(N_1^2 + N_2^2)|^\alpha \to 0.$$

We have, with positive constants c,

$$\frac{1}{n^2} \sum_{t \neq s} E|\mathrm{Cov}(G_t, G_s)(N_1^2 + N_2^2)|^\alpha$$
$$= \frac{1}{n^2} \sum_{t \neq s} E\left| \left(r_1^2 \int_{-\pi}^{x} \cos(\lambda(t-s)) g^2(\lambda)\,d\lambda \right.\right.$$
$$\left.\left. + r_2^2 \int_{-\pi}^{y} \cos(\lambda(t-s)) g^2(\lambda)\,d\lambda \right) (N_1^2 + N_2^2) \right|^\alpha$$
$$\leq \frac{c}{n^2} \sum_{1 \leq t < n} (n-t) \left| r_1^2 \int_{-\pi}^{x} \cos(\lambda t) g^2(\lambda)\,d\lambda + r_2^2 \int_{-\pi}^{y} \cos(\lambda t) g^2(\lambda)\,d\lambda \right|^\alpha$$
$$\leq \frac{c}{n} \sum_{1 \leq t < n} \left(\left| \int_{-\pi}^{x} \cos(\lambda t) g^2(\lambda)\,d\lambda \right|^\alpha + \left| \int_{-\pi}^{y} \cos(\lambda t) g^2(\lambda)\,d\lambda \right|^\alpha \right).$$

We restrict ourselves to show that the term with x converges to 0. In order to avoid a notational mess, we will also restrict ourselves to an even function $g^2(\lambda)$. Using the representation of $g^2(\lambda)$ as a Fourier series with absolutely summable Fourier coeffcients, we obtain, for some constant $c > 0$,

$$\begin{aligned}
&\frac{1}{n}\sum_{1\le t<n}\left|\int_{-\pi}^{x}\cos(\lambda t)g^2(\lambda)\,d\lambda\right|^{\alpha}\\
&\quad=\frac{1}{n}\sum_{1\le t<n}\left|\int_{-\pi}^{x}\cos(\lambda t)\sum_{l=0}^{\infty}\cos(\lambda l)k_l\,d\lambda\right|^{\alpha}\\
&\quad=\frac{1}{n}\sum_{1\le t<n}\left|\frac{1}{2}\sum_{l=0}^{\infty}\int_{-\pi}^{x}k_l(\cos(\lambda(t-l))+\cos(\lambda(t+l)))\,d\lambda\right|^{\alpha}\\
&\quad=\frac{1}{n}\sum_{1\le t<n}\left|\frac{1}{2}\left(\sum_{l=0}^{\infty}k_l\frac{\sin(x(t+l))}{t+l}+\sum_{l=0}^{t-1}k_l\frac{\sin(x(t-l))}{t-l}+k_t(x+\pi)\right.\right.\\
&\qquad\qquad\qquad\qquad\left.\left.+\sum_{l=t+1}^{\infty}k_l\frac{\sin(x(t-l))}{t-l}\right)\right|^{\alpha}\\
&\quad\le\frac{c}{n}\sum_{1\le t<n}\left(\left|\sum_{l=0}^{\infty}\frac{|k_l|}{t+l}\right|^{\alpha}+|k_t|^{\alpha}+\left|\sum_{l=1}^{t}\frac{|k_{t-l}|}{l}\right|^{\alpha}+\left|\sum_{l=1}^{\infty}\frac{|k_{t+l}|}{l}\right|^{\alpha}\right)\\
&\quad\to 0.
\end{aligned}$$

Hence

$$n^{-1}\sum_{t=1}^{n}|G_t|^{\alpha}\to_P E|G_1|^{\alpha}=E|N_1|^{\alpha}\left(r_1^2\int_{-\pi}^{x}g^2(\lambda)\,d\lambda+r_2^2\int_{-\pi}^{y}g^2(\lambda)\,d\lambda+r_3^2\right)^{\alpha/2}$$

for a standard Gaussian r.v. N_1.

Thus we proved for nonnegative r_i that

$$\begin{aligned}
&E\exp\{-r_1M_n(x)-r_2M_n(y)-r_3\gamma_{n,Z}^2\}\\
&\quad\to\exp\left\{-E|\sqrt{2}N_1|^{\alpha}\sigma\left(r_1\int_{-\pi}^{x}g^2(\lambda)\,d\lambda+r_2\int_{-\pi}^{y}g^2(\lambda)\,d\lambda+r_3\right)^{\alpha/2}\right\}\\
&\quad=\exp\left\{-Y_0\left(r_1\int_{-\pi}^{x}g^2(\lambda)\,d\lambda+r_2\int_{-\pi}^{y}g^2(\lambda)\,d\lambda+r_3\right)\right\},
\end{aligned}$$

showing the convergence of the two-dimensional distributions in (6.1). This proves the fidi-convergence in (3.1).

Furthermore, $(\gamma^2_{n,X})$ and the processes $(\int_{-\pi}^{\cdot} \widetilde{I}_{n,X}(\lambda) f(\lambda)\,d\lambda)$ are tight and imply the tightness of the processes $(\int_{-\pi}^{\cdot} I_{n,X}(\lambda) f(\lambda)\,d\lambda)$. This proves the theorem. □

PROOF OF THEOREM 3.2. We use an idea due to Grenander and Rosenblatt (1957), page 189. First we prove the statement for $f(\cdot) = |\psi(\cdot)|^{-2}$. In view of Lemmas 5.1 to 5.3 we have

$$\begin{aligned}
&\left(\gamma^2_{n,X}, T_n, x_n \int_{-\pi}^{\cdot} \left(\frac{I_{n,X}(\lambda)}{|\psi(\lambda)|^2} - T_n\right) d\lambda\right) \\
&\quad = \Bigl(\gamma^2_{n,Z}\psi^2(1+o_P(1)), \gamma^2_{n,Z}(1+o_P(1)), \\
&\qquad\quad x_n \int_{-\pi}^{\cdot} (I_{n,Z}(\lambda) - \gamma^2_{n,Z})\,d\lambda + o_P(1)\Bigr).
\end{aligned} \tag{6.2}$$

Hence it suffices to show that

$$\left(\gamma^2_{n,Z}, x_n \int_{-\pi}^{\cdot} (I_{n,Z}(\lambda) - \gamma^2_{n,Z})\,d\lambda\right) \to_d \left(Y_0, 2\sum_{t=1}^{\infty} Z_t \frac{\sin(\cdot t)}{t}\right).$$

We have

$$x_n \int_{-\pi}^{\cdot} (I_{n,Z}(\lambda) - \gamma^2_{n,Z})\,d\lambda = 2\sum_{t=1}^{n-1} \gamma_{n,Z}(t) \frac{\sin(\cdot t)}{t}.$$

From Proposition 2.1 and the continuous mapping theorem, we conclude that for m fixed

$$\left(\gamma^2_{n,Z}, \sum_{t=1}^{m} \gamma_{n,Z}(t) \frac{\sin(\cdot t)}{t}\right) \to_d \left(Y_0, \sum_{t=1}^{m} Z_t \frac{\sin(\cdot t)}{t}\right).$$

Thus it suffices to show that

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left(\sup_{-\pi\le x\le \pi} \left| \sum_{n>t>m} \gamma_{n,Z}(t) \frac{\sin(xt)}{t}\right| > \varepsilon\right) = 0, \qquad \varepsilon > 0. \tag{6.3}$$

Put

$$\varepsilon_p = 2^{-2p/\gamma}$$

for some $\gamma > 0$ to be chosen later. For notational ease we suppose that $m = 2^a - 1$ and $n = 2^{b+1} - 1$ for integers $a < b$. If m or n do not have this representation, the terms below which correspond to m and n have to be treated separately, but the estimates obtained are completely analogous and therefore omitted.

We have

$$
\begin{aligned}
I &= P\bigg(\sup_{-\pi\le x\le\pi}\bigg|\sum_{n>t>m}\gamma_{n,Z}(t)\frac{\sin(xt)}{t}\bigg|>\varepsilon\bigg)\\
&\le P\bigg(\sum_{p=a}^{b}\sup_{-\pi\le x\le\pi}\bigg|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\frac{\sin(xt)}{t}\bigg|>\varepsilon\bigg)\\
&\le P\bigg(\sum_{p=a}^{b}\varepsilon_p>\varepsilon\bigg)\\
&\quad+P\bigg(\bigcup_{p=a}^{b}\bigg\{\sup_{-\pi\le x\le\pi}\bigg|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\frac{\sin(xt)}{t}\bigg|>\varepsilon_p\bigg\}\bigg)\qquad(6.4)\\
&\le P\bigg(\sum_{p=a}^{b}\varepsilon_p>\varepsilon\bigg)\\
&\quad+\sum_{p=a}^{b}P\bigg(\sup_{-\pi\le x\le\pi}\bigg|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\frac{\sin(xt)}{t}\bigg|>\varepsilon_p\bigg)\\
&= I_1+I_2.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
J_p &= P\bigg(\sup_{0\le x\le\pi}\bigg|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\frac{\sin(xt)}{t}\bigg|>\varepsilon_p\bigg)\\
&= P\bigg(\max_{k=1,\dots,2^p}\max_{j\in\{(k-1)2^p+1,\dots,k2^p\}}\sup_{x\in((j-1)\pi2^{-2p},\,j\pi2^{-2p}]}\bigg|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\frac{\sin(xt)}{t}\bigg|>\varepsilon_p\bigg)\\
&\le \sum_{k=1,\dots,2^p}P\bigg(\max_{j\in\{(k-1)2^p+1,\dots,k2^p\}}\sup_{x\in((j-1)\pi2^{-2p},\,j\pi2^{-2p}]}\bigg|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\frac{\sin(xt)}{t}\bigg|>\varepsilon_p\bigg)\\
&= \sum_{k=1,\dots,2^p}J_{pk}.
\end{aligned}
$$

We have

$$
\begin{aligned}
J_{pk} = P\bigg(&\max_{j\in\{(k-1)2^p+1,\dots,k2^p\}}\sup_{x\in((j-1)\pi2^{-2p},\,j\pi2^{-2p}]}\bigg|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\\
&\times\bigg(\frac{\sin((x-(j-1)\pi2^{-2p})t)\cos((j-1)\pi2^{-2p}t)}{t}\\
&+\frac{\cos((x-(j-1)\pi2^{-2p})t)\sin((j-1)\pi2^{-2p}t)}{t}\bigg)\bigg|>\varepsilon_p\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= P\biggl(\max_{j\in\{(k-1)2^p+1,\dots,k2^p\}} \sup_{0\le x\le 2^{-2p}\pi}\biggl|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\\
&\qquad\qquad \times \frac{\sin(xt)\cos((j-1)\pi 2^{-2p}t)+\cos(xt)\sin((j-1)\pi 2^{-2p}t)}{t}\biggr| > \varepsilon_p\biggr)\\
&\le P\biggl(\max_{j\in\{(k-1)2^p+1,\dots,k2^p\}} \sup_{0\le x\le 2^{-2p}\pi}\biggl|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\\
&\qquad\qquad\qquad\qquad \times \frac{\sin(xt)\cos((j-1)\pi 2^{-2p}t)}{t}\biggr| > \varepsilon_p/2\biggr)\\
&\quad + P\biggl(\max_{j\in\{(k-1)2^p+1,\dots,k2^p\}} \sup_{0\le x\le 2^{-2p}\pi}\biggl|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\\
&\qquad\qquad\qquad\qquad \times \frac{\cos(xt)\sin((j-1)\pi 2^{-2p}t)}{t}\biggr| > \varepsilon_p/2\biggr)\\
&= J_{pk1} + J_{pk2}.
\end{aligned}
$$

We restrict ourselves to the estimation of J_{pk2}. We write $\|\cdot\|$ instead of $\sup_{0\le x\le 2^{-2p}\pi}|\cdot|$ and

$$
Y_j(\cdot) = \sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\frac{\cos(\cdot t)\sin((j-1)\pi 2^{-2p}t)}{t}.
$$

We intend to apply a maximal inequality to

$$
J_{pk2} = P\Bigl(\max_{j\in\{(k-1)2^p+1,\dots,k2^p\}}\|Y_j\| > \varepsilon_p/2\Bigr).
$$

Note that for $j < j'$ both taken from $\{(k-1)2^p+1,\dots,k2^p\}$ and for positive constants c,

$$
\begin{aligned}
&P(\|Y_j - Y_{j'}\| > \varepsilon_p/2)\\
&\quad = P\biggl(\biggl\|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\frac{\cos(\cdot t)}{t}\biggl(2\sin\biggl(\frac{j-j'}{2}\pi 2^{-2p}t\biggr)\\
&\qquad\qquad\qquad \times\cos\biggl(\frac{j+j'-2}{2}\pi 2^{-2p}t)\biggr)\biggr\| > \varepsilon_p/2\biggr)\\
&\quad \le cP\biggl(\biggl\|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\frac{\cos(\cdot t)}{t}\sin\biggl(\frac{j-j'}{2}\pi 2^{-2p}t\biggr)\biggr\| > \varepsilon_p c\biggr).
\end{aligned}
$$

In the last step we used the contraction principle for random quadratic forms [see Kwapien and Woyczynski (1992), Proposition 6.3.2]. For $x < y$ both taken from $[0, 2^{-2p}\pi]$ we have in view of Theorem 3.1 in Rosinski and Woyczynski

(1987) or Theorem 6.9.4 in Kwapien and Woyczynski (1992) that

$$
\begin{aligned}
P\biggl(\biggl|\sum_{t=2^p}^{2^{p+1}-1}\gamma_{n,Z}(t)\biggl(\frac{\cos(yt)}{t}-\frac{\cos(xt)}{t}\biggr)\sin\biggl(\frac{j-j'}{2}\pi 2^{-2p}t\biggr)\biggr| > \varepsilon_p c\biggr) \\
\le c\varepsilon_p^{-\mu}\sum_{t=2^p}^{2^{p+1}-1}\biggl|\biggl(\frac{\cos(yt)}{t}-\frac{\cos(xt)}{t}\biggr)\sin(\frac{j-j'}{2}\pi 2^{-2p}t)\biggr|^{\mu} \\
\le c\varepsilon_p^{-\mu}|x-y|^{\mu}\,|(j-j')2^{-2p}|^{\mu}\,2^{p(\mu+1)}
\end{aligned}
$$

for an arbitrary $1 < \mu < \alpha$ and positive constants c. An application of the maximal inequality Theorem 12.2 in Billingsley (1968) yields that

$$
\begin{aligned}
P(\|Y_j - Y_{j'}\| > \varepsilon_p/2) &\le c\varepsilon_p^{-\mu}2^{-2p\mu}|(j-j')2^{-2p}|^{\mu}2^{p(\mu+1)} \\
&= c|(j-j')2^{-2p}|^{\mu}2^{p(1+\mu(-1+2/\gamma))}
\end{aligned}
$$

and, again applying the same inequality, we obtain

$$
\begin{aligned}
J_{pk2} &= P\Bigl(\max_{j\in\{(k-1)2^p+1,\dots,k2^p\}}\|Y_j\| > \varepsilon_p/2\Bigr) \\
&\le c2^{-p\mu}2^{p(1+\mu(-1+2/\gamma))} = c2^{p(1+2\mu(-1+1/\gamma))}.
\end{aligned}
$$

[Note that Theorem 12.2 in Billingsley (1968) is proved for real-valued Y_j but a careful study of the proof shows that it remains valid for Banach space valued r.v.'s.] Hence we have

$$
J_p \le \sum_{k=1,\dots,2^p} J_{pk} \le c2^{2p(1+\mu(-1+1/\gamma))}.
$$

Now choose γ such that $\kappa = -(1+\mu(-1+1/\gamma)) > 0$. From (6.4) we conclude that for every $\varepsilon > 0$ and a sufficiently large

$$
\limsup_{n\to\infty} I = \limsup_{n\to\infty} I_2 \le c\sum_{p=a}^{\infty}2^{-2p\kappa} \le c2^{-2a\kappa}.
$$

This proves (6.3) and implies (3.3) for $g \equiv 1$.

For general g the relation (3.3) follows by integration by parts and by the continuous mapping theorem. □

The proof of Proposition 3.5 is essentially based on the proof of Lemma 9.3 in Mikosch, Gadrich, Klüppelberg and Adler (1995); it is reformulated as a result for quadratic forms in iid α-stable r.v.'s.

LEMMA 6.1. *Let f_t be real numbers such that*

$$
\sum_{t=-\infty}^{\infty}|f_t|^{\mu} < \infty \tag{6.5}
$$

for some $\mu < \alpha$. *If* $f_0 = 0$, *then*

$$\Bigl(\gamma_{n,Z}^2, y_n^{-1} \sum_{1 \le t,s \le n} f_{t-s} Z_t Z_s\Bigr) \to_d \Bigl(Y_0, \sum_{t=1}^{\infty} (f_t + f_{-t}) Z_t\Bigr)$$

$$=_d \Bigl(Y_0, Z_1 \Bigl(\sum_{t=1}^{\infty} |f_t + f_{-t}|^\alpha\Bigr)^{1/\alpha}\Bigr).$$

If $f_0 \neq 0$, *then*

$$n^{-2/\alpha} \sum_{1 \le t,\, s \le n} f_{t-s} Z_t Z_s \to_d f_0 Y_0.$$

PROOF OF PROPOSITION 3.5. We have by Lemmas 5.1 to 5.4 that

$$\Bigl(\gamma_{n,X}^2, T_n, x_n \int_{-\pi}^{x} (I_{n,X}(\lambda) - |\psi(\lambda)|^2 T_n) f(\lambda)\, d\lambda\Bigr)$$

$$= \Bigl(\psi^2 \gamma_{n,Z}^2 (1 + o_P(1)), \gamma_{n,Z}^2 (1 + o_P(1)),$$

$$x_n \int_{-\pi}^{x} (I_{n,Z}(\lambda) - \gamma_{n,Z}^2) g(\lambda)\, d\lambda + o_P(1)\Bigr)$$

$$= \Bigl(\psi^2 \gamma_{n,Z}^2 (1 + o_P(1)), \gamma_{n,Z}^2 (1 + o_P(1)),$$

$$2x_n \sum_{t=1}^{n-1} \gamma_{n,Z}(t) \int_{-\pi}^{x} \cos(\lambda t) g(\lambda)\, d\lambda + o_P(1)\Bigr).$$

Now we write

$$f_t = \int_{-\pi}^{x} \cos(\lambda t) g(\lambda)\, d\lambda$$

and apply Lemma 6.1 which concludes the proof. □

As Proposition 3.5 shows, the statement of Theorem 3.2 can remain valid for certain frequencies, but not uniformly on $[-\pi, \pi]$. We demonstrate this with the process

$$x_n \int_{-\pi}^{\cdot} \Bigl(\frac{I_{n,X}(\lambda)}{|\psi(\lambda)|^2} - T_n\Bigr) \cos\lambda\, d\lambda \tag{6.6}$$

for a positive power transfer function $|\psi(\cdot)|^2$ satisfying (2.1). By (6.2) the problem of weak convergence of (6.6) can be reduced to the question of weak convergence of

$$D_n(\cdot) = x_n \int_{-\pi}^{\cdot} (I_{n,Z}(\lambda) - \gamma_{n,Z}^2) \cos\lambda\, d\lambda.$$

Given that D_n converges weakly in $C[-\pi, \pi]$ to a limit process of the form

$$2\sum_{t=1}^{\infty} Z_t \int_{-\pi}^{\cdot} \cos\lambda \cos(\lambda t)\, d\lambda =: \sum_{t=1}^{\infty} Z_t f_t(\cdot),$$

a necessary condition for its existence is that

$$\sum_t |f_t(x)|^\alpha < \infty$$

for all $x \in [-\pi, \pi]$. This series converges for $x = -\pi, 0, \pi/2, -\pi/2, \pi$ but it does not for any other rational or irrational multiple of π. Hence for the centering sequence (T_n) weak convergence does not hold in the uniform topology for $\alpha \in (0, 1]$.

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DEPARTMENT OF MATHEMATICS
JOHANNES GUTENBERG UNIVERSITY
D-55099 MAINZ
GERMANY

FACULTY OF MATHEMATICS
AND PHYSICAL SCIENCES
UNIVERSITY OF GRONINGEN
P.O. BOX 800
GRONINGEN
THE NETHERLANDS